Chapter VI

The Quantum Enveloping Algebra of $\mathfrak{sl}(2)$

The aim of Chapters VI–VII is to construct a Hopf algebra $U_q = U_q(\mathfrak{sl}(2))$ which is a one-parameter deformation of the enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$ investigated in Chapter V, and which is in duality with the Hopf algebra $SL_q(2)$ defined in Chapter IV. It will be our second main example of a quantum group. When the parameter q is not a root of unity, the algebra U_q has properties parallel to those of the enveloping algebra of $\mathfrak{sl}(2)$. In the present chapter we classify the simple finite-dimensional modules of U_q and determine its centre. We close the chapter with a few considerations on the case when q is a root of unity.

We assume throughout this chapter that the ground field k is the field of complex numbers.

VI.1 The Algebra $U_q(\mathfrak{sl}(2))$

Let us fix an invertible element q of k different from 1 and -1 so that the fraction $\frac{1}{q-q^{-1}}$ is well-defined. We introduce some notation.

For any integer n, set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}. \tag{1.1}$$

These q-analogues are more symmetric than the ones defined in IV.2, as shown by the relations

$$[-n] = -[n] \quad \text{and} \quad [m+n] = q^n[m] + q^{-m}[n]. \tag{1.2}$$

Observe that, if q is not a root of unity, then $[n] \neq 0$ for any non-zero integer. This is not so when q is a root of unity. In that case, denote by d its order, i.e., the smallest integer > 1 such that $q^d = 1$. Since we assume $q^2 \neq 1$, we must have $d > 2$. Define also

$$e = \begin{cases} d & \text{if } d \text{ is odd} \\ d/2 & \text{when } d \text{ is even.} \end{cases} \tag{1.3}$$

Let us agree that $d = e = \infty$ when q is not a root of unity. Now it is easy to check that

$$[n] = 0 \Longleftrightarrow n \equiv 0 \text{ modulo } e. \tag{1.4}$$

We also have the following versions of factorials and binomial coefficients. For integers $0 \leq k \leq n$, set $[0]! = 1$,

$$[k]! = [1][2]\dots[k] \tag{1.5}$$

if $k > 0$, and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}. \tag{1.6}$$

These q-analogues are related to those of IV.2 by

$$[n] = q^{-(n-1)}\,(n)_{q^2}, \quad [n]! = q^{-n(n-1)/2}(n)!_{q^2}, \tag{1.7}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{-k(n-k)} \begin{pmatrix} n \\ k \end{pmatrix}_{q^2}. \tag{1.8}$$

With this new notation we can rewrite Proposition IV.2.2 as follows. If x and y are variables subject to the relation $yx = q^2xy$, then we have $(n > 0)$

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}. \tag{1.9}$$

Definition VI.1.1. *We define $U_q = U_q(\mathfrak{sl}(2))$ as the algebra generated by the four variables E, F, K, K^{-1} with the relations*

$$KK^{-1} = K^{-1}K = 1, \tag{1.10}$$

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \tag{1.11}$$

and

$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{1.12}$$

The rest of the section is devoted to a few elementary properties of U_q. The following lemma has an easy proof left to the reader.

Lemma VI.1.2. *There is a unique algebra automorphism of U_q such that*

$$\omega(E) = F, \quad \omega(F) = E, \quad \omega(K) = K^{-1}.$$

The automorphism ω is sometimes called the *Cartan automorphism.* We now state a q-analogue of Lemma V.3.1.

Lemma VI.1.3. *Let $m \geq 0$ and $n \in \mathbf{Z}$. The following relations hold in U_q:*

$$E^m K^n = q^{-2mn}\, K^n E^m, \quad F^m K^n = q^{2mn}\, K^n F^m,$$

$$\begin{aligned}[E, F^m] &= [m]\, F^{m-1} \frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}} \\ &= [m]\, \frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}} F^{m-1},\end{aligned}$$

$$\begin{aligned}[E^m, F] &= [m]\, \frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}} E^{m-1} \\ &= [m]\, E^{m-1} \frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}.\end{aligned}$$

PROOF. The first two relations result trivially from Relations (1.11). The third one is proved by induction on m using

$$[E, F^m] = [E, F^{m-1}]F + F^{m-1}[E, F] = [E, F^{m-1}]F + F^{m-1}\frac{K - K^{-1}}{q - q^{-1}}$$

as in the proof of Lemma V.3.1. Applying the automorphism ω to the third relation, one gets the fourth one. □

We now describe a basis of U_q by showing that U_q is an iterated Ore extension. We refer to I.7–8 for information concerning Ore extensions.

Proposition VI.1.4. *The algebra U_q is Noetherian and has no zero divisors. The set $\{E^i F^j K^\ell\}_{i,j \in \mathbf{N};\, \ell \in \mathbf{Z}}$ is a basis of U_q.*

PROOF. Define $A_0 = k[K, K^{-1}]$. We shall construct two Ore extensions $A_1 \subset A_2$ such that A_2 is isomorphic to U_q. First, observe that the algebra A_0 has no zero divisors and is Noetherian as a quotient of a (Noetherian) two-variable polynomial algebra. The family $\{K^\ell\}_{\ell \in \mathbf{Z}}$ is a basis of A_0.

Consider the automorphism α_1 of A_0 determined by $\alpha_1(K) = q^2 K$ and the corresponding Ore extension $A_1 = A_0[F, \alpha_1, 0]$: the latter has a basis consisting of the monomials $\{F^j K^\ell\}_{j \in \mathbf{N},\, \ell \in \mathbf{Z}}$. An argument analogous to the one used to prove Lemma IV.4.2 shows that A_1 is the algebra generated by F, K, K^{-1} and the relation $FK = q^2 KF$.

We now build an Ore extension $A_2 = A_1[E, \alpha_1, \delta]$ from an automorphism α_1 and an α_1-derivation of A_1. The automorphism α_1 is defined by

$$\alpha_1(F^j K^\ell) = q^{-2\ell} F^j K^\ell. \tag{1.13}$$

Let us take as given for a moment that there exists an α_1-derivation δ such that

$$\delta(F) = \frac{K - K^{-1}}{q - q^{-1}} \quad \text{and} \quad \delta(K) = 0.$$

Then the following relations hold in A_2:

$$EK = \alpha_1(K)E + \delta(K) = q^{-2}KE$$

and

$$EF = \alpha_1(F)E + \delta(F) = FE + \frac{K - K^{-1}}{q - q^{-1}}.$$

From these one easily concludes that A_2 is isomorphic to U_q. It then results from Corollary I.7.2 and from Theorem I.8.3 that U_q has the required properties. □

It remains to prove the following technical lemma in order to complete the proof of Proposition 1.4.

Lemma VI.1.5. *Denote by $\delta(F)(K)$ the Laurent polynomial $\frac{K-K^{-1}}{q-q^{-1}}$, and set $\delta(K^\ell) = 0$ and*

$$\delta(F^j K^\ell) = \sum_{i=0}^{j-1} F^{j-1}\delta(F)(q^{-2i}K)K^\ell \tag{1.14}$$

when $j > 0$. Then δ extends to an α_1-derivation of A_1.

PROOF. We must check that, for all $j, m \in \mathbf{N}$ and all $\ell, n \in \mathbf{Z}$, we have

$$\delta(F^j K^\ell \cdot F^m K^n) = \alpha_1(F^j K^\ell)\delta(F^m K^n) + \delta(F^j K^\ell)F^m K^n. \tag{1.15}$$

Let us compute the right-hand side of (1.15) using (1.11), (1.13), and (1.14). We have

$$\begin{aligned}
&\alpha_1(F^j K^\ell)\delta(F^m K^n) + \delta(F^j K^\ell)F^m K^n \\
&= \sum_{i=0}^{m-1} q^{-2\ell}\, F^j K^\ell F^{m-1}\delta(F)(q^{-2i}K)K^n \\
&\qquad + \sum_{i=0}^{j-1} F^{j-1}\delta(F)(q^{-2i}K)K^\ell F^m K^n
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{m-1} q^{-2\ell-2\ell(m-1)}\, F^{j+m-1}\delta(F)(q^{-2i}K)K^{\ell+n} \\
&\qquad + \sum_{i=0}^{j-1} q^{-2\ell m}\, F^{m+j-1}\delta(F)(q^{-2i-2m}K)K^{\ell+n} \\
&= \sum_{i=0}^{m-1} q^{-2\ell m}\, F^{m+j-1}\delta(F)(q^{-2i}K)K^{\ell+n} \\
&\qquad + \sum_{i=m}^{j+m-1} q^{-2\ell m}\, F^{m+j-1}\delta(F)(q^{-2i}K)K^{\ell+n} \\
&= q^{-2\ell m}\Big(\sum_{i=0}^{j+m-1} F^{j+m-1}\delta(F)(q^{-2i}K)K^{\ell+n}\Big) \\
&= q^{-2\ell m}\delta(F^{j+m}K^{\ell+n}) \\
&= \delta(F^jK^\ell \cdot F^mK^n).
\end{aligned}
$$

□

VI.2 Relationship with the Enveloping Algebra of $\mathfrak{sl}(2)$

One expects to recover $U = U(\mathfrak{sl}(2))$ from U_q by setting $q = 1$. This is impossible with Definition 1.1. So we first have to give another presentation for U_q.

Proposition VI.2.1. *The algebra U_q is isomorphic to the algebra U'_q generated by the five variables E, F, K, K^{-1}, L and the relations*

$$KK^{-1} = K^{-1}K = 1, \tag{2.1}$$

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \tag{2.2}$$

$$[E, F] = L, \quad (q - q^{-1})L = K - K^{-1}, \tag{2.3}$$

$$[L, E] = q(EK + K^{-1}E), \quad [L, F] = -q^{-1}(FK + K^{-1}F). \tag{2.4}$$

Observe that, contrary to U_q, the algebra U'_q is defined for all values of the parameter q, in particular for $q = 1$. In some sense, it would have been better to proceed through the whole theory of the quantum enveloping algebra of $\mathfrak{sl}(2)$ with U'_q rather than with U_q, but the simpler presentation given in Section 1 is sufficient for our purposes.

PROOF. Set

$$\varphi(E) = E, \quad \varphi(F) = F, \quad \varphi(K) = K$$

and

$$\psi(E) = E, \quad \psi(F) = F, \quad \psi(K) = K, \quad \psi(L) = [E, F].$$

It is clear that φ gives rise to a well-defined morphism of algebras from U_q to U'_q. Let us show that $\psi : U'_q \to U_q$ is well-defined too. It suffices to check that the images under ψ of the defining Relations (2.1) hold in the algebra U_q. This is clearly true for Relations (2.1–2.2) and for $[E, F] = L$. For the remaining relation in (2.3) we have

$$(q - q^{-1})\psi(L) = (q - q^{-1})[E, F] = K - K^{-1}.$$

For the first relation in (2.4) we get

$$\begin{aligned}[\psi(L), \psi(E)] = [[E, F], E] &= \frac{1}{q - q^{-1}}[K - K^{-1}, E] \\ &= \frac{(q^2 - 1)EK + (q^2 - 1)K^{-1}E}{q - q^{-1}} \\ &= q\,(EK + K^{-1}E).\end{aligned}$$

One derives the last relation in a similar fashion.

The reader may now verify that φ and ψ are reciprocal algebra morphisms by checking the necessary relations on the generators. □

The relationship with the enveloping algebra U is given in the following statement.

Proposition VI.2.2. *If $q = 1$, we have*

$$U'_1 \cong U[K]/(K^2 - 1) \quad \textit{and} \quad U \cong U'_1/(K - 1).$$

PROOF. It suffices to prove the first isomorphism. Now U'_1 has the following presentation: it is generated by E, F, K, K^{-1}, L and Relations (2.1–2.4) in which q has been replaced by 1, namely

$$KK^{-1} = K^{-1}K = 1, \tag{2.5}$$

$$KEK^{-1} = E, \quad KFK^{-1} = F, \tag{2.6}$$

$$[E, F] = L, \quad K - K^{-1} = 0, \tag{2.7}$$

$$[L, E] = (EK + K^{-1}E), \quad [L, F] = -(FK + K^{-1}F). \tag{2.8}$$

Relations (2.5–2.6) imply that K is central. Relation (2.7) yields $K^2 = 1$, which allows one to rewrite the Relations (2.8) as

$$[L, E] = 2EK, \quad [L, F] = -2FK. \tag{2.9}$$

We then get an isomorphism from U'_1 to $U[K]/(K^2 - 1)$ by sending E to XK, F to Y, K to K, and L to HK. □

In particular, the projection of U'_1 onto U is obtained by sending E to X, F to Y, K to 1, and L to H. One may use this projection to rederive certain relations in U (for instance, Lemma V.3.1) from their q-analogues in U'_q.

VI.3 Representations of U_q

We assume in this section that the complex parameter q is not a root of unity. Our aim is to determine all finite-dimensional simple U_q-modules under this assumption by closely following the methods of Section V.4.

For any U_q-module V and any scalar $\lambda \neq 0$, we denote by V^λ the subspace of all vectors v in V such that $Kv = \lambda v$. The scalar λ is called a *weight* of V if $V^\lambda \neq \{0\}$.

Lemma VI.3.1. *We have $EV^\lambda \subset V^{q^2\lambda}$ and $FV^\lambda \subset V^{q^{-2}\lambda}$.*

PROOF. For $v \in V^\lambda$ we have

$$K(Ev) = q^2 E(Kv) = q^2\lambda\, Ev \quad \text{and} \quad K(Fv) = q^{-2}F(Kv) = q^{-2}\lambda\, Fv.$$

□

Definition VI.3.2. *Let V be a U_q-module and λ be a scalar. An element $v \neq 0$ of V is a highest weight vector of weight λ if $Ev = 0$ and if $Kv = \lambda v$. A U_q-module is a highest weight module of highest weight λ if it is generated by a highest weight vector of weight λ.*

Proposition VI.3.3. *Any non-zero finite-dimensional U_q-module V contains a highest weight vector. Moreover, the endomorphisms induced by E and F on V are nilpotent.*

PROOF. Since $k = \mathbf{C}$ is algebraically closed and V is finite-dimensional, there exists a non-zero vector w and a scalar α such that $Kw = \alpha w$. If $Ew = 0$, the vector w is a highest weight vector and we are done. If not, let us consider the sequence of vectors $E^n w$ where n runs over the non-negative integers. According to Lemma 3.1, it is a sequence of eigenvectors with distinct eigenvalues; consequently, there exists an integer n such that $E^n w \neq 0$ and $E^{n+1}w = 0$. The vector $E^n w$ is a highest weight vector.

In order to show that the action of E on V is nilpotent, it suffices to check that 0 is the only possible eigenvalue of E. Now, if v is a non-zero eigenvector for E with eigenvalue $\lambda \neq 0$, then so is $K^n v$ with eigenvalue $q^{-2n}\lambda$. The endomorphism E would then have infinitely many distinct eigenvalues, which is impossible. The same argument works for F. □

Lemma VI.3.4. *Let v be a highest weight vector of weight λ. Set $v_0 = v$ and $v_p = \frac{1}{[p]!} F^p v$ for $p > 0$. Then*

$$Kv_p = \lambda q^{-2p}\, v_p, \quad Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}}\, v_{p-1}, \quad Fv_{p-1} = [p]\, v_p.$$

PROOF. These relations result from Lemma 1.3. □

We now determine all finite-dimensional simple U_q-modules.

Theorem VI.3.5. (a) *Let V be a finite-dimensional U_q-module generated by a highest weight vector v of weight λ. Then*

(i) *The scalar λ is of the form $\lambda = \varepsilon\, q^n$ where $\varepsilon = \pm 1$ and n is the integer defined by* $\dim(V) = n + 1$.

(ii) *Setting $v_p = F^p v/[p]!$, we have $v_p = 0$ for $p > n$ and, in addition, the set $\{v = v_0, v_1, \dots, v_n\}$ is a basis of V.*

(iii) *The operator K acting on V is diagonalizable with the $(n+1)$ distinct eigenvalues $\{\varepsilon q^n, \varepsilon q^{n-2}, \dots, \varepsilon q^{-n+2}, \varepsilon q^{-n}\}$.*

(iv) *Any other highest weight vector in V is a scalar multiple of v and is of weight λ.*

(v) *The module V is simple.*

(b) *Any simple finite-dimensional U_q-module is generated by a highest weight vector. Two finite-dimensional U-modules generated by highest weight vectors of the same weight are isomorphic.*

Proof. (a) According to Lemma 3.4, the sequence $\{v_p\}_{p\geq 0}$ is a sequence of eigenvectors for K with distinct eigenvalues. Since V is finite-dimensional, there has to exist an integer n such that $v_n \neq 0$ and $v_{n+1} = 0$. The formulas of Lemma 3.4 then show that $v_m = 0$ for all $m > n$ and $v_m \neq 0$ for all $m \leq n$. By Lemma 3.4, we also have

$$0 = Ev_{n+1} = \frac{q^{-n}\lambda - q^n\lambda^{-1}}{q - q^{-1}} v_n.$$

Hence, $q^{-n}\lambda = q^n\lambda^{-1}$, which is equivalent to $\lambda = \pm q^n$. The rest of the proof of (i)–(iii) is as in the classical case (see Theorem V.4.4).

(iv) Let v' be another highest weight vector. It is an eigenvector for the action of K; hence, it is a scalar multiple of some vector v_i. But, again by Lemma 3.4, the vector v_i is killed by E if and only $i = 0$.

(v) Let V' be a non-zero U_q-submodule of V and let v' be a highest weight vector of V'. Then v' also is a highest weight vector for V. By (iv), v' has to be a non-zero scalar multiple of v. Therefore v is in V'. Since v generates V, we must have $V \subset V'$, which proves that V is simple.

(b) The proof is the same as for Theorem V.4.4 (b). □

Theorem 3.5 implies that, up to isomorphism, there exists a unique simple U_q-module of dimension $n+1$ and generated by a highest weight vector of weight εq^n. We denote this module by $V_{\varepsilon,n}$ and the corresponding morphism of algebras $U_q \to \mathrm{End}(V_{\varepsilon,n})$ by $\rho_{\varepsilon,n}$. Observe that the formulas of Lemma 3.4 may be rewritten as follows for $V_{\varepsilon,n}$:

$$Kv_p = \varepsilon\, q^{n-2p}\, v_p, \tag{3.1}$$

$$Ev_p = \varepsilon\, [n - p + 1]\, v_{p-1}, \tag{3.2}$$

and

$$Fv_{p-1} = [p]\, v_p. \tag{3.3}$$

As a special case, we have $V_{\varepsilon,0} = k$. The morphism $\rho_{\varepsilon,0}$ is given by

$$\rho_{\varepsilon,0}(K) = \varepsilon, \quad \rho_{\varepsilon,0}(E) = \rho_{\varepsilon,0}(F) = 0.$$

We shall see in VII.1 that $\rho_{\varepsilon,0}$ may be identified with the counit of a Hopf algebra structure on U_q. It will imply that the module $V_{1,0}$ is trivial and that any trivial U_q-module is isomorphic to a direct sum of copies of $V_{1,0}$. On the other hand, the module $V_{-1,0}$ is not trivial.

On the $(n+1)$-dimensional module $V_{\varepsilon,n}$, the generators E, F and K act by operators that can be represented on the basis $\{v_0, v_1, \ldots, v_n\}$ by the matrices

$$\rho_{\varepsilon,n}(E) = \varepsilon \begin{pmatrix} 0 & [n] & 0 & \cdots & 0 \\ 0 & 0 & [n-1] & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\rho_{\varepsilon,n}(F) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & [2] & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & [n] & 0 \end{pmatrix},$$

and

$$\rho_{\varepsilon,n}(K) = \varepsilon \begin{pmatrix} q^n & 0 & \cdots & 0 & 0 \\ 0 & q^{n-2} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q^{-n+2} & 0 \\ 0 & 0 & \cdots & 0 & q^{-n} \end{pmatrix}.$$

So far, we have built U_q-modules generated by highest weight vectors whose weights λ had special values. Let us now show that there exist highest weight modules with arbitrary highest weights.

Let us fix a scalar $\lambda \neq 0$. Consider an infinite-dimensional vector space $V(\lambda)$ with denumerable basis $\{v_i\}_{i\in\mathbf{N}}$. For $p \geq 0$, set

$$Kv_p = \lambda q^{-2p}\, v_p, \quad K^{-1}v_p = \lambda^{-1}q^{2p}\, v_p, \tag{3.4}$$

$$Ev_{p+1} = \frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}}\, v_p, \quad Fv_p = [p+1]\, v_{p+1} \tag{3.5}$$

and $Ev_0 = 0$.

Lemma VI.3.6. *Relations* (3.4–3.5) *define a U_q-module structure on $V(\lambda)$. The element v_0 generates $V(\lambda)$ as a U_q-module and is a highest weight vector of weight λ.*

PROOF. Immediate computations yield

$$\begin{aligned} KK^{-1}v_p &= v_p, & K^{-1}Kv_p &= v_p \\ KEK^{-1}v_p &= q^2Ev_p, & KFK^{-1}v_p &= q^{-2}Fv_p. \end{aligned}$$

We also have

$$\begin{aligned} [E,F]v_p &= \left([p+1]\,\frac{q^{-p}\lambda - q^{p}\lambda^{-1}}{q-q^{-1}} - [p]\,\frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}\right) v_p \\ &= \frac{q^{-2p}\lambda - q^{2p}\lambda^{-1}}{q-q^{-1}}\, v_p \\ &= \frac{K-K^{-1}}{q-q^{-1}} v_p. \end{aligned}$$

This proves that Relations (3.4–3.5) define a U_q-module structure on $V(\lambda)$.

Next, we have $Kv_0 = \lambda\, v_0$ and $Ev_0 = 0$, which means that v_0 is a highest weight vector of weight λ. Finally, (3.5) implies that $v_p = F^p v_0/[p]!$ for all p, which proves that $V(\lambda)$ is generated by v_0. □

By analogy with the classical case, the highest weight U_q-module $V(\lambda)$ is called the *Verma module* of highest weight λ. It enjoys the following universal property.

Proposition VI.3.7. *Any highest weight U_q-module V of highest weight λ is a quotient of the Verma module $V(\lambda)$.*

PROOF. Let v be a highest weight vector generating V. We define a linear map f from $V(\lambda)$ to V by $f(v_p) = 1/[p]!\, F^p v$. Lemma 3.4 implies that f is U_q-linear. Since $f(v_0) = v$ generates V, the map f is surjective. □

In particular, the simple finite-dimensional module $V_{\varepsilon,n}$ described above is a quotient of the Verma module $V(\varepsilon q^n)$. As a consequence, the module $V(\lambda)$ cannot be simple when λ is of the form $\pm q^n$ where n is a nonnegative integer.

VI.4 The Harish-Chandra Homomorphism and the Centre of U_q

Our next objective is to describe the centre Z_q of U_q in case q is not a root of unity. We assume this throughout this section.

We start by introducing a special central element of U_q. It is sometimes called the *quantum Casimir element.*